SELECTING AND CREATING MATHEMATICAL TASKS

Extracts from*: Selecting and Creating Mathematical Tasks: From Research to Practice,* Margaret Schwan Smith and Mary Kay Stein 1998*; Mathematical Task Analysis Guide,* National Council of Teachers of Mathematics

What features of a mathematics classroom really make a difference in how students come to view mathematics and what they ultimately learn? Is it where the students are working in small groups? Is it where the students are using manipulatives? Is it the nature of the of the mathematical tasks that are given to students? Research conducted in the QUASAR project, a five-year study of mathematics education reform in middle schools, offers some insight into these questions.

A major finding of this research is that the highest learning gains on a mathematics performance assessment were related to the extent to which tasks were set up and implemented in ways that engage students in high levels of cognitive thinking and reasoning. This finding supports the position that the nature of the tasks to which students are exposed determines what students learn, and this leads to many questions that should be considered by middle school teachers.

In particular, results by Stein and Lane suggest the importance of starting with high level, cognitively complex tasks if the ultimate goal is to have students develop the capacity to think, reason, and problem solve. Selecting and setting up high level tasks does not guarantee students engagement at a high level. Starting with a good task does, however, appear to be a necessary condition, since low level tasks almost never result in high level engagement.

How do you know a good task when you see one?

When classifying a mathematical task as "good" that is, as having the potential to engage students in high level thinking, we first considered the students - their age, grade level, prior knowledge and experiences - and the norms and expectations for work in their classroom. For example, a task in which students are asked to add five 2-digit numbers and explain the process they used. For a 5th or 6th grade student who has access to a calculator, the addition algorithm, or both ,and for whom "explain the process" means "tell how you did it", the task could be considered routine. If however, the task is given to a second grader who has just started work with 2-digit numbers, who has base-10 pieces available, and for whom "explain the process" means you means "you need to explain your thinking" the task may indeed be high level. Therefore, when a teacher selects a task for use in their classroom ll the factors need to be considered to determine the extent to which a task is likely to afford an appropriate level of challenge for his or her students.

A second step for classifying tasks as "good" is to consider the four categories of cognitive demand described in Stein and Smith (1998) in the table on the next page:

- Memorisation
- Procedures without connections to concepts or meanings
- Procedures with connections to concepts and meanings
- Doing mathematics

Using these categories as templates we ask ourselves what kind of thinking a task will demand of the students. Tasks where students perform a memorised procedure in the routine manner lead to one level of thinking; tasks that ask students to think conceptually lead to a very different set of thinking processes.

Mathematical tasks may have different features - for instance some might require an explanation or a description, some can be solved using manipulatives, some have real world contexts, some involve multiple steps or actions or judgments, some make use of diagrams. We need to be careful when looking at the tasks to look beyond the superficial features to focus on the kind of thinking in which students must engage to complete the tasks.

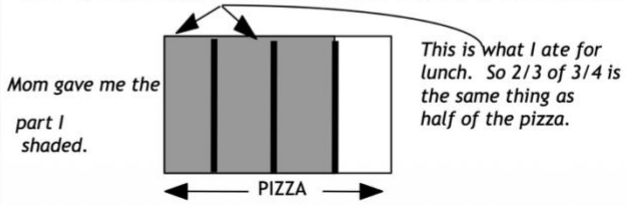
The Mathematical Task Analysis Guide (TAG)

Lower-Level Demands	Higher-Level Demands
Memorization Tasks • Involves either producing previously learned facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory. • Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • Are not ambiguous – such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated. • Have no connection to the concepts or meaning that underlie the facts, rules, formulae, or definitions being learned or reproduced.	**Procedures With Connections Tasks** • Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
Procedures Without Connections Tasks • Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task. • Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • Have no connection to the concepts or meaning that underlie the procedure being used. • Are focused on producing correct answers rather than developing mathematical understanding. • Require no explanations, or explanations that focus solely on describing the procedure that was used.	**Doing Mathematics Tasks** • Requires complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). • Requires students to explore and to understand the nature of mathematical concepts, processes, or relationships. • Demands self-monitoring or self-regulation of one's own cognitive processes. • Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • Requires students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • Requires considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

Stein M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). Implementing standards-based mathematics instruction: A casebook for professional development, p. 16. New York: Teachers College Press.

On the next page you will find some examples of tasks categorised under this system. How would your students perform on these given tasks? How could these classifications be helpful in selecting and creating worthwhile mathematics tasks for use in your own classroom? What other issues might be important to raise in a discussion of tasks?

The Four Levels of Cognitive Demand of Multiplication of Fractions Tasks

Lower-Level Demands	Higher-Level Demands
Memorization What is the rule for multiplying fractions? *Expected Student Response:* *You multiply the numerator times the numerator and the denominator times the denominator.* OR *You multiply the two top numbers and then the two bottom numbers.*	Procedures With Connections Find 1/6 of 1/2. Use pattern blocks. Draw your answer and explain your solution. *Expected Student Response:* *First you take half of the whole which would be one hexagon. Then you take one-sixth of the half. So I divided the hexagon into six pieces which would be six triangles. I only needed one-sixth so that would be one triangle. Then I needed to figure out what part of the two hexagons one triangle was and it was 1 out of 12. So 1/6 of 1/2 is 1/12.*
Procedures Without Connections Multiply: $\frac{1}{6} \times \frac{1}{2}$ $\frac{2}{3} \times \frac{3}{4}$ $\frac{4}{9} \times \frac{3}{5}$ *Expected Student Response:* $\frac{1}{6} \times \frac{1}{2} = \frac{1 \times 1}{6 \times 2} = \frac{1}{12}$ $\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$ $\frac{4}{9} \times \frac{3}{5} = \frac{4 \times 3}{9 \times 5} = \frac{12}{45}$	Doing Mathematics Create a real-world situation for the following problem: $\frac{2}{3} \times \frac{3}{4}$. Solve the problem you have created without using the rule and explain your solution. *One Possible Student Response:* *For lunch Mom gave me three-fourths of the pizza that we ordered. I could only finish two-thirds of what she gave me. How much of the whole pizza did I eat?* *I drew a rectangle to show the whole pizza. Then I cut it into fourths and shaded three of them to show the part mom gave me. Since I only ate two thirds of what she gave me, that would be only two of the shaded sections.* 

There is one more factor we need to consider when looking when categorising tasks: what role is the teacher taking and what role is the student taking? Who is doing the thinking? It is certainly true that the same task can be presented very differently in two different classrooms. The role that the teacher takes and the way that they support and question their students is critical for understanding the level of cognitive demand expected of students ,and hence the outcomes that the students are going to achieve.

The following examples, using the same stimulus, "Seth's Birthday Cake" is illustrative of this. On the four-level of mathematical task demands, where would you place each of the three examples on the next page?

Seth's Birthday Cake

After his birthday party, Seth had one-fifth of his cake leftover. Eight of his friends spent the night. His friends plan to eat the remaining cake as a snack later that night, and they each expect to receive an equal share of the remaining cake. They want to know what fractional part of the original cake will each get to eat for a snack.

A. What fractional part of the cake did each friend eat for a snack? Draw and clearly label a diagram that shows how much of the cake is eaten by one person for the snack. Use words to explain your reasoning and decisions for how you drew the diagram.

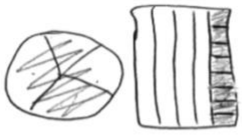
B. Seth claims that to find the answer you can use the expression $8 \div 1/5$. His friend Frankie says that the expression would be $1/5 \div 8$. Who is correct? Why?

Adapted from *Multiplication of Fractions,* Institute for Learning, University of Pittsburgh, 2015 (ifl.pitt.edu).

Figure 9.2. A task on division contexts involving fractions

Scenario 1. Nhia

Nhia first drew a circle, and then crossed it out on her paper. She finally seemed to settle on the drawing to the right for the birthday cake task.



Teacher:	Tell me about your drawing?
Nhia:	I think this one might work. *(Points to* the *diagram on the right.)* This is the cake and I cut it into fifths. Then I cut one of those fifths into eight pieces.
Teacher:	So, why did you cut it into eight pieces?
Nhia:	Well, there were 8 friends and they shared that leftover piece so I cut it into eighths.
Teacher:	So now you have to describe the size of the piece that each friend got.
Nhia:	Yeah, but I don't get that. Is the answer one-eighth?
Teacher:	Remember when we did a problem like this the other day, and we had to make sure the whole was all cut into equal sized pieces. What is the whole in this problem?
Nhia:	The cake.
Teacher:	So right now you have some big pieces and some small pieces. How many small pieces would you have it you took all five of the big pieces and broke each one into eight smaller pieces just like you did for that one part already?
Nhia:	40?

Scenario 3: Zhen

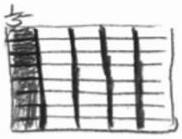
As the teacher approaches Zhen, he is reading and rereading the problem, but has not yet put anything on his paper.

Zhen:	I don't get it.
Teacher:	What are you suppose to figure out?
Zhen:	How to share the cake with 8 friends.
Teacher:	Okay, how might you start?
Zhen:	I don't know. They only got a fifth.
Teacher:	Okay, the first thing you need to do is draw a rectangle for the cake and then cut it into fifths because there was one fifth of the cake left. Then next you need to cut those pieces into 8 smaller pieces each for the 8 friends.
Zhen:	Make eighths?
Teacher:	No, cut the cake into fifths. Do that much first. Then I'll help you with what to do next.
Zhen:	Oh. *(The student draws a rectangle and partitions it into fifths.)*
Teacher:	Good, now shade one fifth to show how much cake was left.
Zhen:	*(The student shades one fifth.)*
Teacher:	Good. Now cut the other way across the cake to get 8 pieces. How many small pieces are in the whole cake?
Zhen:	40.
Teacher:	Okay, so if each friend got one of those small pieces, what fraction of the whole cake did each friend get?
Zhen:	One-fortieth?
Teacher:	Yes, each friend got one-fortieth. Now go on the next part of the task.

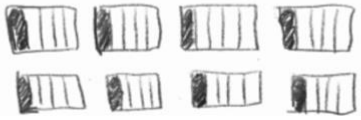
Scenario 4. Diego

Diego determines that each friend gets one-fortieth of the cake and then claims that both equations $8 \div 1/5$ and $1/5 \div 8$ work because the order does not matter with the commutative property.

Teacher:	Tell me about your thinking, why do you think both could be right?
Diego:	They're both right because you can just switch the numbers and you get the same answer.
Teacher:	Interesting, so tell me what you did here with your picture and why. *(Teacher points to the student's diagram.)*



Diego:	I made a rectangle for the cake and cut it into fifths cuz that's what they had left. Then I cut it into eighths 'cuz there were 8 friends sharing it. And each friend got one-fortieth.
Teacher:	So, did what you just describe sound more like $8 \div 1/5$ or more like $1/5 \div 8$?
Diego:	Well…um… 8 friends shared that piece, I'm thinking it's more like $1/5 \div 8$.
Teacher:	You said, you thought you would get the same answer with either expression, so tell me about $8 \div 1/5$ and how that would be the same.
Diego:	I could make 8 rectangles (*draws the picture below*) and cut each into fifths (*subdivides each rectangle*). Each person gets one piece (*shades one part of each rectangle*)... mmm ... I got 40 pieces, but…now I'm not sure, is each piece one-fifth or one-fortieth?



Teacher:	What do you think?
Diego:	I think this is different this would be 8 cakes cut into fifths, and I got 40 fifths, not fortieths.
Teacher:	So are the expressions the same or different?
Diego:	No, they're different. I guess you can't just switch them. Frankie is right it's $1/5 \div 8$.