MATHEMATICAL CONNECTIONS AND THE BEAUTY OF MATHEMATICS

EXTRACTS FROM: "MAKING MATHEMATICAL CONNECTIONS", LINDA M GOJAK, NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, OCTOBER 3, 2013 **AND** "WHAT IS MATHEMATICAL BEAUTY? TEACHING THROUGH BIG IDEAS AND CONNECTIONS", JO BOALER, YOUCUBED

In this article two respected mathematicians discuss the importance of understanding big ideas in mathematics, and how this leads to a deeper understanding of mathematics through enabling students to make connections between topics.

Linda M. Gojak, NTMC President, writes:

One of the most memorable moments I had in teaching mathematics occurred in a fifth-grade class. We began the year using rectangular arrays as a model to develop the concept of prime and composite numbers. We hung student-made posters of the numbers from 1 to 100 with representations of arrays and lists of factors for each number around the room. By the end of that unit all my students had mastered multiplication facts and could factor with facility as we began our work with fractions. The connections among concepts and the use of concrete representations certainly lead to deeper understanding.

Later that year students worked with a variety of models to find area and perimeter of rectangles and extended that experience to find the areas of triangles, parallelograms and trapezoids. Most students were able to generalise a formula, albeit not always the most efficient, for each polygon. One day a student commented that this was just like what they had studied at the beginning of the year. When I gave a puzzled look, the class pointed to the posters still on the wall from our first unit of study and said, "You know that factor and multiple stuff." I had a new appreciation for the power of providing experiences that enable students to make connections among mathematical ideas. My students remembered and understood the mathematics that we had studied months earlier!

Since that experience I have thought about progressions and trajectories of content. Connections have continued to pique my interest. I believe that instructional programmes from pre-kindergarten through grade 12 should enable student should enable all students to:

- recognise and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognise and apply mathematics in context outside of mathematics.

Too often, rather than making sense of mathematical ideas, students focus on remembering procedures or tricks. For example, how many students learn "flip and multiply" to divide fractions - but have no idea why it works? Often those who understand why the procedure works struggle to apply it in problem situations. The procedure alone leads to misconceptions. Students who work from rote memory often invert the wrong fraction, forget to change operations, or even apply the rule when multiplying two fractions.

The meaning of operations doesn't change from whole numbers to fractions. For example, in the early grades, the understanding that students develop of division of whole numbers often rests on the idea that "9 divided by 3 for example, asks how many groups of 3 are in 9. As students move to fractions, it is important to provide them with experiences that connect this whole number understanding to similar examples with fractions "9 / 16 divided by 3 / 16 for example asks how many groups of 3 sixteenths are in 9 sixteenths. In this way, students gain a deeper understanding rather than depending on a memorised procedure, and can apply division of fractions to a variety of problem-solving situations and real-world applications.

Many teachers use manipulative materials to introduce a new concept. Manipulatives themselves, however, do not ensure understanding. We must provide experiences that support students' efforts to make connections between what they're doing with the materials and the mathematical ideas they represent. This takes time and

teacher expertise. Algebra tiles are not an end to teaching basic algebra concepts - when used appropriately, they provide students with opportunities to connect their work to the concepts. And it is these connections that enable students to make sense of the abstract representations.

Although it is important to think about the connections among concepts within the grade level or courses that we teach, it is also important to reflect on the connections across grade levels. This work involves thoughtful discussions with other colleagues about the way that concepts are taught and the potential linkages among these ideas. Many of us learned mathematics as isolated pieces of information. Taking a mathematical concept and considering how it originates, extends, and connects with other concepts across the grades will help teachers develop a deeper understanding. It is then that we can plan instruction that ensures that our students regularly make connections to help them make sense of the mathematics they are learning.

Jo Boaler, Youcubed, writes:

Mathematics is a beautiful subject. Ask mathematicians and others what they love about the subject and they will talk about the amazing connections that thread through the terrain, unifying the different ideas. There are not many facts or methods to remember in mathematics, but there are a few really big and important ideas that are connected to each other and that infuse the subject. Yet when we ask students what they think maths is, most will say that it's a lot of different rules and methods. This is really unfortunate as students who believe mathematics is a set of methods to be remembered are the lowest achieving students, worldwide, as revealed by PISA data.

So why do so few students, or teachers, see mathematics as a set of rich ideas and connections? One of the reasons is that teachers are given sets of standards to teach and no matter how good the standard writers are, they all cut mathematics up into small pieces and give teachers small atomised content areas - usually a set of methods - to teach. The connections disappear - teachers cannot see them and they are lost from students learning pathways. Instead, teachers see the list of content - often 100 or more methods in a year - and work systematically through them. This too often leads teachers to skim through content quickly as when mathematics is disconnected and offered in small sections there is a lot to get through in any year. The connections between ideas are invisible and students do not develop one of the most important insights into mathematics that they can develop - that mathematics is a set of connected, big ideas.

An important step in seeing mathematical connections is knowing the big ideas. I have systematically studied different school approaches in different countries for many years. In two longitudinal experimental studies of students learning mathematics in middle and high school, one in the UK and one in the US, I followed students through different teaching approaches. In both cases the students who achieved at higher levels and who enjoyed mathematics more had learned mathematics using more conceptual approaches and the teachers had planned the teaching around big ideas.

In the UK, the successful teachers used a project-based approach. They decided on the big ideas in the different years of mathematics and then chose long, open-ended projects through which the students would encounter a need for different mathematical methods. One of the projects students worked on was called, “Interpreting the World”. Students collected different forms of data about the world that they chose, and analysed. At the end of three years of this approach the students scored significantly higher levels on national mathematics exams, than students who worked systematically through method after method in textbook studies. Studies of the students six years later showed that students who learnt through a project-based approach ended up in more professional jobs, than those who learn traditionally, which they attributed to the ways they had learned mathematics.

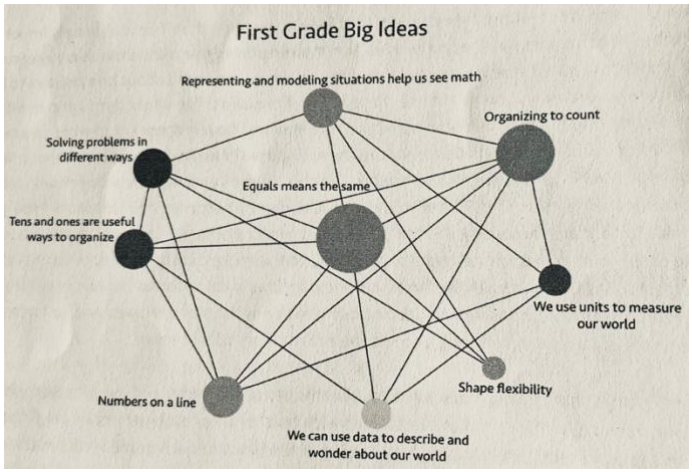
The teaching of methods to students when they encounter a need for them has been shown in research to be the most effective method for teaching mathematics (Schwartz and Bransford ,1998). Students can learn through activities focused upon big ideas and when they counter a need for a new method they learn it within the activity.

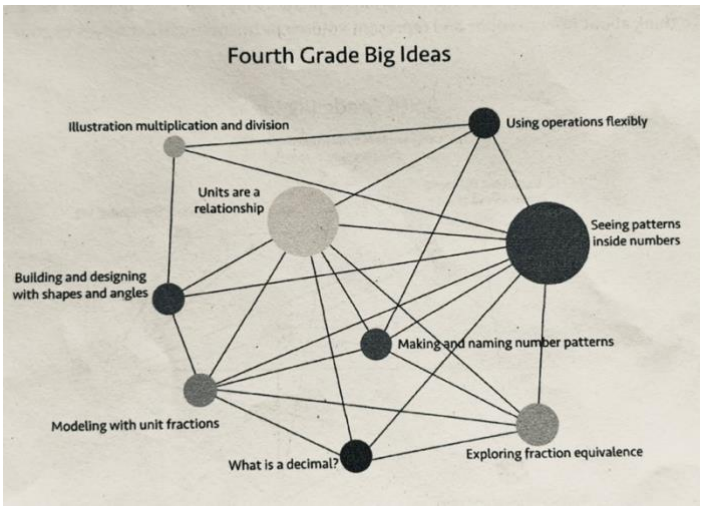
When students do this their brains are primed to learn the new method as they are curious and they need the method. When teaching through the big ideas in mathematics most of the smaller ideas and methods naturally arise and teachers can teach them in a meaningful purposeful way. The ideas that never come up are probably not very important to learn!

In our work teachers frequently ask us - what are the big ideas? Many teachers themselves were taught mathematics as the list of content, missing understanding of the big ideas that make mathematics a cohesive whole. It is ideal for teachers to sit down and work out big unifying ideas together. When teachers work on identifying and discussing big ideas, they become attuned to the mathematics that is most important and that they may see in tasks. They also develop greater appreciation for the connections that run between the tasks and ideas.

My team mapped the big ideas and connections in the Common Core Standards in the US curriculum. We began by looking at the standards and considered what connections existed within and across them that build on ideas for a previous grade and were pivotal to ideas in later grades. We also thought carefully about the ideas that get little attention in standards and curriculum common, but are powerful for mathematical thinkers, and we included these ideas in our materials. We tested our big ideas by creating the networks that follow, knowing that if an idea is truly important it will be connected to other ideas in the grade level. These connections give mathematics coherence which supports all students in making sense, as students draw on what they know about one big idea to learn about another. Two examples of this mapping of networks are shown below:

FIRST GRADE: Organising and seeing equivalence are ideas which pervade first grade. Students need to develop ways to organise for counting and comparing, and ultimately make meaning out of our place value system. Equivalence means learning to assess what makes things different and the same. For instance, 4 plus 1 and 5 are equivalents, even though they look different, and students may develop a dozen strategies for adding 4 and 1 to arrive at 5. Those strategies are different, but related and equivalent in the result they produce. Grappling with equivalence and organisation is important work in first grade.





FOURTH GRADE: Patterning and examining relationships are at the heart of 4th grade. Students begin to think about how to identify and express patterns, both visually and numerically, and build foundations for proportional reasoning when thinking about the connections between units. Students look within fractions and decimals for the relationships represented there - relationships between numerator and denominators, fraction and decimal, and decimal and place value. Fourth graders use relationships to connect multiplication and division and think flexibly across all operations.

PUTTING IT INTO PRACTICE: There are several ways you might use these maps of big ideas. You might get together with colleagues and discuss how you understand the meaning of each big idea and how it connects to what you teach. What opportunity do you or could students have to explore these big ideas? How could you make these big ideas more prominent in your teaching?

You might also explore the connections within each network, and ask, how is this big idea connected to the others? How would you describe that connection? How could you give the students the opportunity to connect these ideas? Finally, you might explore how the big ideas in one grade level are connected to those in previous or later grades.

As we created these networks of big ideas, it was clear to us that there are some even bigger ideas that pervade all of mathematics, eg just looking for and naming patterns; examining and naming relationships; composing and decomposing with numbers and shapes; and so forth. Looking across grade levels is useful way to begin to see the full mathematical terrain.

Our hope in providing these big ideas is that they will encourage thinking about the threads that weave across mathematics and that are pivotable in students' mathematics learning. Many teachers, as well as students, have been given the faulty idea that mathematics is just a long list of methods and disconnected rules. We hope these ideas will initiate rich conversations between teachers and the big ideas and the connections that relate them to each other. When teachers are thinking about big ideas they become more attuned to the most important tasks to choose for students, and the connections to highlight in classroom discussions. Looking at the big ideas in different grades will also help teachers to see what students have been learning and will be learning.

We are now in the 21st century. Students do not need to be trained to be calculators - we have technology for this - but they do need to experience mathematics as a beautiful and connected subject of enduring big ideas. Students who learn through bigger ideas and connections enjoy mathematics more, understand mathematics more deeply, and are better prepared to tackle the big complex problems and discoveries that they will meet in their lives.