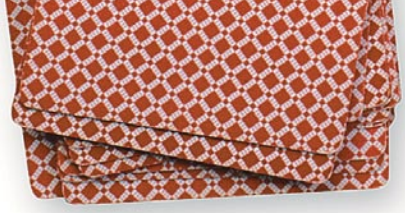
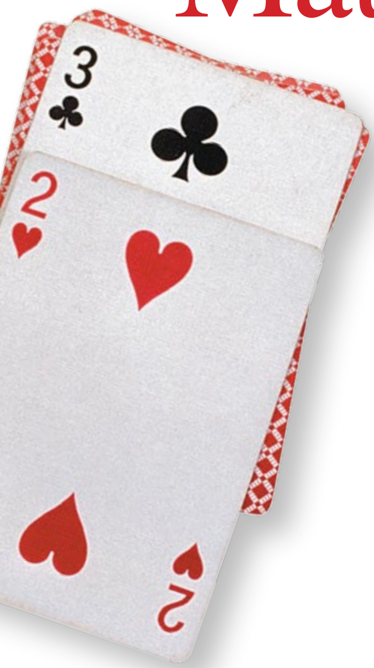


Developing Students' Mathematical Reasoning through Games



"Do you have a six?" Gina asked Martin.

"Yes, I do," replied Martin as he passed his card to Gina.

"So I have two cards that make ten," said Gina excitedly, showing her six of hearts and a four of spades.

"Gina, do you have a five?" asked Carl.

"No," replied Gina, and Carl drew a card.

People of all ages love to play games. This game, Make 10, was created for first-grade students to help them find combinations of numbers with a sum of 10 and to give them practice with number facts (see **fig. 1**). The children work together to make number pairs using a deck of cards. A team "wins" if it makes the most pairs. Adults recognize that all the teams will have the same number combinations (in other words, no group is going to discover something unique), but this is not obvious to children. Many times ideas that are obvious to adults can launch a mathematical discussion in an elementary school classroom. In this case, just ask the children why the teams had the same number combinations and why they think they all found the same pairs. Good games for the classroom are engaging and create opportunities for students to explore mathematical ideas.

Kindergarteners and first graders enjoy games based on chance. These games provide opportunities to explore fundamental number concepts such as the counting sequence, one-to-one correspondence, and quantity. Third- and fourth-grade students are intrigued by games of strategy, which require players to consider multiple options, predict future moves, and plan a series of moves that will maximize their chance of winning. Engaging mathematical games encourage students to explore number combinations, place value, patterns, and other important mathematical concepts. The three *P*s—*plan, play,* and *please be patient*—provide a framework to help teachers consider a game's potential for exploring mathematical ideas with students and leading to a rich discussion.

By Jo Clay Olson

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The Three *P*s

When choosing a game for the classroom, first play the game yourself with a family member or a colleague to gain familiarity with its rules and subtleties. Discuss the mathematical ideas embedded in the game and how these ideas may emerge during play. Determine the level of competition appropriate for your students and decide whether the rules need to be modified to meet their needs. Anticipate some possible responses or strategies that your students may use while playing the game. From your list of anticipated responses, create a list of questions that you can ask to probe your students' thinking during play.

Plan

Plan how you will introduce the game to your class. Will two students demonstrate while you explain? Will you invite one student to play the game with you? Or will you play against the whole class? Decide on an appropriate amount of class time to devote to playing the game. Remember that playing a game for the first time requires a period of learning and clarification. As students become more familiar with the game, they will spend less time learning the rules and more time exploring mathematical ideas.

Decide how you will pair students or form the small groups who then start the game on their own. Students who use less sophisticated strategies should be partnered with students who use slightly

more mature strategies but not with the most abstract thinkers. Abstract thinkers and students who use immature strategies may be unable to communicate with each other because they conceptualize mathematical ideas in very different ways (Carpenter et al. 1999). The slightly more mature thinkers can also be paired with the abstract thinkers because they use language with common meanings. As they explain their solutions or strategy, the more mature thinkers develop their articulation of mathematical ideas and thus lead less mature thinkers to more abstract ideas.

Play

Introduce the game and then, while the students play, walk around the room and listen to their conversations. Ask probing questions and listen to the students' responses. Rather than restate a student's strategy, ask his or her partner to explain it. Take notes to record the different strategies that your students use and to plan the class discussion. Decide which strategies to discuss first, beginning with less mature ideas and then moving to more sophisticated strategies. Ask the students how the strategies are similar and how they are different. Possible discussion starters include these:

- "Thumbs up if you liked the game, thumbs sideways if it was okay, and thumbs down if you didn't like it. What did you like about it? Why?"
- "What did you notice while playing the game?"
- "Did you make any choices while playing?"
- "Did anyone figure out a way to quickly find a solution?"

After the discussion, ask the students to partner with a different person to explain their strategy and then give them five minutes to write the alternative strategy in their mathematics journal. Last, suggest that the next time they play the game they use a different strategy.

Please be patient

Provide repeated opportunities for your students to play the game, and let the mathematical ideas emerge as they notice new patterns, relationships, and strategies. Watch carefully how students incor-

Figure 1

Rules for playing the game Make 10

Make 10

Goal		Make number pairs with a sum of 10.
Players		Game is played by two or three children.
Rules	1.	Shuffle a deck of cards with the numbers 0 to 10 or 1 to 9.
	2.	Deal five cards to each player. Place the remaining cards face down on the table.
	3.	Player 1 asks one of the other players for a card to add to one of her or his cards to make a sum of 10. The requested card is then placed with a second card from player 1's hand, and the other players check the sum. If the player does not have the requested card, player 1 draws one card from the face-down stack. If player 1 can make a sum of 10 with two cards, the pair is placed on the table.
	4.	The players draw additional cards from the face-down stack until they each have five cards. If player 1 cannot make a sum of 10 with the cards in her or his hand, player 1 keeps the six cards and does not draw additional cards until he or she has fewer than five cards.
	5.	The game is over when the face-down cards have been used up. The students count the number of pairs that they made, and the group with the largest number wins. (Note: Because the game should result in every group finding the same number of pairs, everyone should win. This outcome can prompt a rich discussion as to why this is the case.)
Modifications	1.	Play the game competitively; each player tries to get the greatest number of pairs.
	2.	Allow students to use two or three cards to make a sum of 10.
	3.	Change the goal from making 10 to creating the largest two-digit number (this game is called Double Digits).
	4.	Change the goal from making 10 to making the highest sum with two cards (this game is called Super Sums).

porate more abstract strategies into their own. Be patient and allow the mathematical ideas to develop over time. Your patience empowers students to independently explore mathematical ideas and create conceptual understandings that they will not forget. To illustrate the three *P*s framework for using games in the classroom, three games are presented here along with a brief discussion of primary and intermediate students' developmental needs while playing games.

A Game for Primary-Level Students: Close to 20

Games that five-year-olds like to play are generally based on luck: With each card a player draws or with each roll of a die, the player has only one choice. Board games, card games, and games with dice allow primary-level students to explore fundamental number concepts by counting and using one-to-one correspondence as they move a game piece on a board (Fosnot and Dolk 2001). Eventually, the children notice that larger numbers in the counting sequence yield a greater number of spaces on the game board, and they begin to conceptualize two aspects of numbers: Numbers represent both an element in the counting sequence and a quantity.

First and second graders enjoy using mathematical reasoning to play more sophisticated games. Competition should be minimized to keep their thinking focused on the mathematical ideas. The game Close to 20 (Akers et al. 1997) promotes mathematical thinking while offering practice in number facts. I was introduced to this game while team teaching with a first-grade teacher. The different strategies that the children used to find number combinations close to 20 intrigued me, and the game quickly became a classroom favorite.

Close to 20 is played by using a set of cards numbered 0 to 9 and recording sheets. Each player is dealt five cards and then chooses three of the five cards to make a sum as close to 20 as possible (but not more than 20). A player's score for the round is the difference between his or her combination and 20. On the recording sheet (**fig. 2**), each player writes his or her combination, the total of the combination, and the score for the round (20 – total). At the end of the round, each player discards the cards used to make this first combination and receives three new cards. Play continues for five rounds, when the game is over. To minimize competition and enhance cooperation, have each student pair or student group aim to have the lowest combined total in the class.

Plan

When I played this game, I quickly realized that I was not finding all the possible sums close to 20. So I tried some different strategies. One strategy I used was to pick the largest card of the five I was dealt and then pick two more cards whose sum was greater than 10. I knew that for the sum of my three cards to be 20, the sum of these two cards had to be equal to the difference between my largest card and 10. Another strategy I used was based on my knowledge that 3 times 6 is 18 (a number close to 20 but still less than 20). Thus, I would select three cards that were close to the number 6 (e.g., 6, 5, and 7).

The mathematical ideas underlying Close to 20 include these:

- Numbers can be combined in different ways.
- The closest sum may be greater than 20.
- Estimation and related facts can reduce the number of combinations you have to check for the closest sum.

During the second week of school, I wanted to assess whether a group of second graders recognized that numbers could be combined in several ways to make a larger number and to ascertain their knowledge of number facts. Close to 20 was a game I could use both as a formative assessment and as a prompt for a mathematical discussion. I decided to minimize competition by encouraging the students to help one another find sums close to 20 and making the objective a lower score.

Play

To introduce the game, I played the game with another student.

Teacher. We are going to play a card game called Close to 20. What do you think that we're going to try to do?

Yasmeen. I know—we can use cards to make 20.

Teacher. Do you think that we'll always be able to make 20?

Marquis. Yes. You can make lots of numbers with cards.

Larry. No. … Sometimes you get only little cards and can't make a big number.

Teacher. I wonder what you will find out. Who wants to play the game with me? (*Hands went up*.) Okay, Luis, come sit here. Please deal me five cards and then give yourself five cards. Now, let's look at my cards. Who can help me make 20 with three cards? *(We used 5, 6, and 7 to make 18.)* Now, here is my recording sheet. What should we write down?

Marquis. Put the numbers on the lines and write your sum here (*pointing at the space next to the equals symbol*).

Teacher. What should we do here? (*I pointed at the last column*.)

Shaundra. Put 2 in there, you need 2 more to get to 20.

Teacher. Now let's look at Luis's cards. (*Luis shared his cards and possible combinations. Most of the students were ready to play, and I gave cards and two recording sheets to pairs of children.*)

While the second graders played the game, I walked around the room. I noticed that they recorded the first combination that they created. When I had demonstrated the game, several children suggested combinations, and we picked the

Figure 2

Recording sheet for the game Close to 20

Close to 20 Recording Sheet

Name ______________________

							Total	Score for Round (20 – total)
______	+	______	+	______	=		______	______
______	+	______	+	______	=		______	______
______	+	______	+	______	=		______	______
______	+	______	+	______	=		______	______
______	+	______	+	______	=		______	______
							Game Total	______

largest one. Clearly, the students were focused on finding only one combination that was less than 20. When I asked the children why they used a particular combination, I heard two responses: "I just picked it" and "I knew that 4 plus 2 plus 5 is less than 20 because they are all little numbers." When I asked how they figured out the difference between their sum and 20, the students replied, "I counted on my fingers." After ten minutes of play, I decided that it was time for us to discuss the game.

I began our discussion by asking, "Do you remember when we talked about the game and we wondered if we could make 20 every time? What happened?" The second graders now voiced their observation that it was hard to get 20. Then I asked, "Did any of you make 20?" The students looked at their sheets, and three children raised their hands. I wrote their combinations on the board and asked the class to look for a strategy that might help them make combinations of 20. My questions and comments helped the second graders think more deeply about different strategies for selecting cards:

- "What do you notice about these combinations?"
- "Oh, they all begin with the largest number. What do you think the children were thinking about?"
- "How did you figure out sums of numbers without counting on your fingers?"
- "How could you use combinations that you know to play the game?"

Please be patient

The second graders played the game for 15 more minutes. To construct sums that were closer to 20, the children first selected the largest number in their hand. They understood that different numbers could be combined to make a sum of 20, but I did not see them trying several sums before recording one. A few children used base-ten blocks, multilink cubes, and hundreds charts as tools to help them find combinations and the difference between their sum and 20. During my observation of their resumed play, I saw children using some of the ideas that we had discussed. I wondered whether calculators would allow them to find the combination that was the closest to 20. Would calculators enhance their exploration of number patterns and relationships, or would they become a tool for finding an answer? I realized that providing calculators might change the focus of the game and that I needed to carefully consider how and when to introduce them.

Cautions

Sometimes teachers use games solely to practice number facts. These games usually do not engage children for long because they are based on memorization. Some children are quick to memorize, while others need a few moments to use a related fact to compute. Children placed in situations in which recall speed determines success may infer that being "smart" in mathematics means getting the correct answer quickly instead of valuing the process of thinking. Consequently, they may feel incompetent when they use number patterns or related facts to arrive at a solution and may begin to dislike mathematics because they are not fast enough.

A Game for Intermediate-Level Students: The Product Game

Students in the intermediate grades are intrigued by games based on strategy and competition. All strategy games provide students with opportunities to make choices about the best strategy to use while allowing them to explore mathematical ideas. Discussion about different strategies creates a context for developing students' conceptual understanding of patterns, related number facts, and place value. Many students use a peer's explanation to support their own learning by modifying it to one that makes sense or comparing the strategies for the "best" one. A reflective question such as "What did you think about when you selected a number pair?" encourages students to articulate their mathematical ideas. Following is a discussion of the Product game to illustrate the three *P*s.

Plan

In the Product game, students use a variety of strategies to decide the best first move. The goal of the game is to claim any four squares in a row or in a column, any four squares that form a diagonal, or the four squares at the corners. **Figure 3** shows a game board designed for fourth-grade students to explore the products that can be made from the factors 0, 1, 2, 3, 4, 5, and 6.

To begin, player 1 places a paper clip on one factor on the game board. Player 2 places a second paper clip on another factor and claims the product of these two factors by placing a colored chip on that square on the game board. Player 1 then moves one of the clips to a different factor and claims the resulting product. The players continue taking turns by moving one paper clip and claiming the resulting product until one player wins—that is, claims four squares—or a draw is declared.

The first time I played the Product game with the factors 0 through 6, I realized that in my mind I was substituting other factor pairs for the products on the game board. I was amazed at the number of facts that I quickly considered while deciding which product to claim. Later, I noted that the factors 0, 1, and 2 were used to generate more products on the game board than the factors 4, 5, and 6 and decided that it was better to claim products smaller than 16. I claimed the larger products when the opportunity arose and when they extended a row, column, or diagonal.

The mathematical ideas underlying the Product game include these:

- A product is generated by a pair of factors.
- Some products can be created by more than one factor pair.
- The game board changes when different factors are used.
- A finite set of factors can be combined to generate a finite number of products.

Play

I usually introduce this game by playing against the class. After briefly explaining the rules, I place one paper clip on a factor on the game board and then select a student to pick another factor. The class claims the resulting square with a colored chip. I move one paper clip to another factor and place a different colored chip on my square. After several moves, I win and challenge the class to one more game. After the second game, the students are ready to pair up and play against each other. While the student pairs play, I engage students in conversations. Following is an excerpt from a discussion that I had with two fourth-grade students.

Teacher. (*I placed a marker on 5, and José claimed the square with the number 10 by placing his marker on 2.*) José, that is an interesting first move. Why did you pick it?

José. There are five ways to win.

Teacher. Cherie, do you see any of the ways that José could win?

Cherie. Well, he could win by making a column, row, or either diagonal. That's only four ways.

José. There are two ways on the column. I could use 5, 10, 16, and 25 or 1, 5, 10, and 16.

Teacher. Interesting. Cherie, what do you think is the best first move?

Cherie. I like to take 5 first.

Teacher. Why?

Cherie. It's easier to get four in a row.

Teacher. Why is it easier?

Cherie. Uhm ...Well, the numbers are smaller and that makes it easier.

Teacher. José, what do you think?

José. There's only three ways that you can, so it isn't as good.

Cherie. But some of the products can be gotten in lots of ways so it's easier to fill them in.

José. What do you mean?

Cherie. See, you can get 4 by either 1 and 4 or putting both clips on the 2. Six is the same way, so it's easier to win.

Teacher. Those are both great strategies. Why don't you try to find out which one is better?

Figure 3

Game board for the Product game for use with the factors 0 through 6

0	1	2	3
4	5	6	8
9	10	Free	12
15	16	18	20
24	25	30	36

Figure 4

Student-constructed 6-by-6 game board for the Product game for use with the factors 1 through 9

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

In this dialogue, the reflective questions I asked prompted José and Cherie to explore two different mathematical ideas. José explored the patterns of rows, columns, and diagonals to determine the best first move. Cherie considered the number of factor pairs that would enable her to fill in the squares. Both used mathematical reasoning to predict possible outcomes for their strategies. Initially I interpreted Cherie's response that she liked the smaller number to indicate knowledge of the products using the factors 1, 2, and 4, but it became clear that she based her strategy on the relationship between factor combinations and products. I encouraged the two students to explore each other's reasoning by investigating which strategy was "better." Investigations based on students' mathematical thinking prompt mathematical discussions among students as they compare the strengths and limitations of different strategies.

Please be patient

These fourth graders enjoyed playing the Product game for many months. We extended the game by using the factors 1 through 9 and created a 6-by-6 game board (see **fig. 4**). The students designed a double-elimination tournament conducted during indoor recess and after school. Students who were eliminated during the tournament watched their peers play against one another, talked about strategies, and designed a round-robin tournament for themselves. I found myself listening to their discussions, impressed with their articulation of mathematical ideas, development of computational fluency, and enthusiasm for learning and using mathematics. Learning was enhanced when I created opportunities for students to explore, reflect, and discuss their mathematical observations by providing time, asking questions, and being patient.

Cautions

Many students enjoy playing games in which the person with the quickest correct response wins and continues to play against other class members—for example, Around the World or Math Baseball. Games of this type pose an ethical question, however: Does the game provide an equal opportunity for all students to gain fluency with number facts? Students who need the most practice with number facts usually sit at their desks while the students who know the facts have more opportunities to practice. Thus, the very students who need to practice have less opportunity to learn. When struggling students have a turn, they do not have time to use mathematical reasoning and often simply guess. Often their responses are incorrect, and they return to their seat, silently hoping to avoid another turn.

Summary

Games are fun and create a context for developing students' mathematical reasoning. They provide opportunities for students to wonder why some peers are quick to respond and thus encourage students to compare different strategies. Through playing and analyzing games, students also gain computational fluency by describing more efficient strategies and discussing relationships among numbers. Teachers can create opportunities for students to explore mathematical ideas by planning questions that prompt students to reflect about their reasoning and make predictions.

Driscoll (1999) suggests that teachers sometimes limit their questions to three types: managing, clarifying, and orienting. Managing questions help students focus on the problem and begin to work. Clarifying questions help students interpret the problem and select a problem-solving strategy that will lead to a correct solution. Orienting questions keep students thinking about the problem and motivate them toward a correct answer. To develop students' conceptual understanding, Driscoll encourages teachers to expand their repertoire of questions to include those that prompt reflection and those that elicit algebraic reasoning.

While the children were playing Close to 20, I prompted their reflection by asking why they used a particular combination and how they figured out the difference between their sum and 20. Using their responses, I planned a discussion that would encourage them to reflect on whether they could always generate a sum of 20. Then I encouraged them to use algebraic reasoning to find a numeric pattern when comparing three solutions that generated a sum of 20. These questions prompted the students to use more sophisticated thinking to generate sums that were closer to 20.

Teachers help students develop algebraic reasoning by asking questions that prompt them to solve problems by using forward and backward processes (Driscoll 1999). For example, in the Product game I prompted forward thinking by asking, "What are the best first moves? Why?" Such questions encourage students to think several moves ahead and identify factors that will generate products that will have them well positioned on the game board. I encouraged backward thinking by asking students to begin with a product and then identify factors that formed the given product. For this game, a good backward-thinking question is, "What if our opponent could win by claiming the square with a 16. What factors would we want to avoid placing a paper clip on?"

As students play games and analyze strategies, they explore mathematical ideas and compare different strategies for efficiency. The three *P*s format can help teachers use games to develop students' conceptual understanding. When we carefully consider the questions we ask and plan an appropriate level of competition, students stay focused on the mathematics instead of on winning.

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