CHAPTER 4

Creating Mathematical Mindsets: The Importance of Flexibility with Numbers

Babies and infants love mathematics. Give babies a set of blocks, and they will build and order them, fascinated by the ways the edges line up. Children will look up at the sky and be delighted by the V formations in which birds fly. Count a set of objects with a young child, move the objects and count them again and they will be enchanted by the fact they still have the same number. Ask children to make patterns in colored blocks and they will work happily making repeating patterns—one of the most mathematical of all acts. Keith Devlin has written a range of books showing strong evidence that we are all natural mathematics users, and thinkers (see, for example, Devlin, 2006). We want to see patterns in the world and to understand the rhythms of the universe. But the joy and fascination young children experience with mathematics are quickly replaced by dread and dislike when they start school mathematics and are introduced to a dry set of methods they think they just have to accept and remember.

In Finland, one of the highest-scoring countries in the world on PISA tests, students do not learn formal mathematics methods until they are seven. In the United States, United Kingdom, and a few other countries, students start much earlier, and by the time our students are seven they have already been introduced to algorithms for adding, subtracting, multiplying, and dividing numbers and been made to memorize multiplication facts. For many students their first experience of math is one of confusion, as the methods do not make sense to them. The inquisitiveness of our children's early years fades away and is replaced by a strong belief that math is all about following instructions and rules.

The best and most important start we can give our students is to encourage them to play with numbers and shapes, thinking about what patterns and ideas they can see. In my previous book I shared the story of Sarah Flannery, who won the Young Scientist of the Year Award for inventing a new mathematical algorithm. In her autobiography she talks about the way she developed her mathematical thinking from working on puzzles in the home with her dad, and how these puzzles were more important to her than all of her years of math class (Flannery, 2002). Successful math users have an approach to math, as well as mathematical understanding, that sets them apart from less successful users. They approach math with the desire to understand it and to think about it, and with the confidence that they can make sense of it. Successful math users search for patterns and relationships and think about connections. They approach math with a *mathematical mindset,* knowing that math is a subject of growth and their role is to learn and think about new ideas. We need to instill this *mathematical mindset* in students from their first experiences of math.

Research has shown definitively the importance of a growth mindset—the belief that intelligence grows and the more you learn, the smarter you get. But to erase math failure we need students to have growth beliefs about themselves and accompany them with growth beliefs about the nature of mathematics and their role in relation to it. Children need to see math as a conceptual, growth subject that they should think about and make sense of. When students see math as a series of short questions, they cannot see the role for their own inner growth and learning. They think that math is a fixed set of methods that either they get or they don't. When students see math as a broad landscape of unexplored puzzles in which they can wander around, asking questions and thinking about relationships, they understand that their role is thinking, sense making, and growing. When students see mathematics as a set of ideas and relationships and their role as one of thinking about the ideas, and making sense of them, they have a mathematical mindset.

Sebastian Thrun, CEO of Udacity and a research professor at Stanford University, has a mathematical mindset. I started working with Sebastian two years ago. Initially I knew him as a computer science professor and the person who invented self-driving cars, taught the first MOOC, and led teams developing Google Glass and Google Maps. Sebastian moved from his highly successful online course, taken by 160,000 people, to forming Udacity, an online learning company. I started working with Sebastian when he asked for my advice on Udacity's courses. Sebastian is a very high-level user of mathematics whose many accomplishments are known around the world. He has written mathematics books that are so complex that they would, as he says, "make your head smoke." What is less well known about Sebastian is that he is highly reflective about the ways he knows and learns mathematics. When I interviewed Sebastian for my online course (How to Learn Math) for teachers and parents, he talked about the important role played by intuition in mathematics learning and problem solving, and of making sense of situations. He gave a specific example of a time when he was developing robots to be used at the Smithsonian Institution and a problem came up. The children and other visitors at the Smithsonian were creating background noise that was confusing the robots. Sebastian said that he and his team had to go back to their drawing boards and construct new mathematical pathways that would solve the problem and allow the robots to function. He eventually solved the problem by using intuition. Sebastian describes the process of working out a mathematical solution that made sense to him intuitively, then going back and proving it using mathematical methods. Sebastian speaks strongly about never going forward in mathematics unless something makes intuitive sense. In my online course he gives advice to children learning mathematics to never work with formulae or methods unless they make sense and to "just stop" if the methods don't make sense.

So how do we develop mathematical mindsets in students so that they are willing to approach math with sense making and intuition? Before they start school, the task is straightforward. It means asking children to play with puzzles, shapes, and numbers and think about their relationships. But in the early years of school we live in a system whereby students are required, from an early age, to learn many formal mathematical methods, such as those used to add, subtract, divide, and multiply numbers. This is the time when students stray from mathematical mindsets and develop fixed, procedural mindsets. This is the time when it is most critical that teachers and parents introduce mathematics as a flexible conceptual subject that is all about thinking and sense making. The domain of early number work gives us the perfect example of the two mindsets that can develop in students, one that is negative and leads to failure and one that is positive and leads to success.

Number Sense

Eddie Gray and David Tall are two British researchers who worked with students, aged 7 to 13, who had been nominated by their teachers as being either low-, middle-, or high-achieving students (Gray & Tall, 1994). All of the students were given number problems, such as adding or subtracting two numbers. The researchers found an important difference between the low- and high-achieving students. The high-achieving students solved the questions by using what is known as number sense—they interacted with the numbers flexibly and conceptually. The low-achieving students used no number sense and seemed to believe that their role was to recall and use a standard method even when this was difficult to do. For example, when students were given a problem such as 21 – 6, the high-achieving students made the problem easier by changing it to 20 –5, but the low-achieving students counted backward, starting at 21 and counting down, which is difficult to do and prone to error. After extensive study of the different strategies that the students used, the researchers concluded that the difference between high- and low-achieving students was not that the low-achieving students knew less mathematics, but that they were interacting with mathematics differently. Instead of approaching numbers with flexibility and number sense, they seemed to cling to formal procedures they had learned, using them very precisely, not abandoning them even when it made sense to do so. The low achievers did not *know less,* they just did not use numbers flexibly—probably because they had been set on the wrong pathway, from an early age, of trying to memorize methods and number facts instead of interacting with numbers flexibly (Boaler, 2015). The researchers pointed out something else important—the mathematics the low achievers were using was a harder mathematics. It is much easier to subtract 5 from 20 than to start at 21 and count down 16 numbers. Unfortunately for low achievers, they are often identified as struggling with math and therefore given more drill and practice—cementing their beliefs that math success means memorizing methods, not understanding and making sense of situations. They are sent down a damaging pathway that makes them cling to formal procedures, and as a result they often face a lifetime of difficulty with mathematics.

A mathematical mindset reflects an active approach to mathematics knowledge, in which students see their role as understanding and sense making. Number sense reflects a deep understanding of mathematics, but it comes about through a mathematical mindset that is focused on making sense of numbers and quantities. It is useful to think about the ways number sense is developed in students, not only because number sense is the foundation for all higher level mathematics (Feikes & Schwingendorf, 2008) but also because number sense and mathematical mindsets develop together, and learning about ways to develop one helps the development of the other.

Mathematics is a conceptual domain. It is not, as many people think, a list of facts and methods to be remembered.

In Figure 4.1, the purple arrows represent methods to be learned; the pink boxes represent the concepts being learned. Starting at the bottom left of the diagram, we see the method of counting. When students learn to count, they remember order and names for numbers, but they also develop the *concept* of number; that is, the idea of a number. In the early stages of learning to add numbers, students learn a method called "counting on." Counting on is used when you have two sets of numbers—for example, 15 plus 4—and you learn to count the first set: counting to 15, then continuing counting: 16–17–18–19. When students learn the method of counting on, they develop the concept of "sum." This is not a method of addition; it is a conceptual idea. In the next stage of their mathematics work, students may learn to add groups of numbers, such as three groups of 4, and as they learn to add groups, they develop the concept of a product. Again, this is not a method (of multiplication); it is a conceptual idea. The ideas of a number,

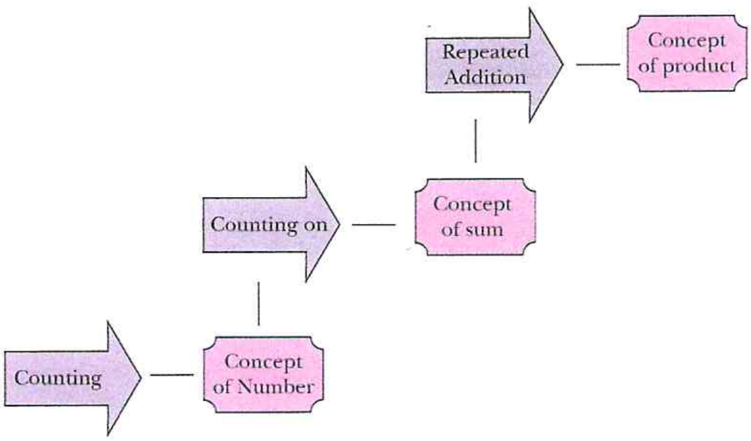


FIGURE 4.1 Mathematics methods and concepts
Source: Gray & Tall, 1994.

a sum, and a product are concepts in mathematics that students need to think deeply about. Students should learn methods, such as adding and multiplying, not as ends in themselves but as part of a conceptual understanding of numbers, sums, and products and how they relate to each other.

We know that when we learn mathematics we engage in a brain process called *compression*. When you learn a new area of mathematics that you know nothing about, it takes up a large space in your brain, as you need to think hard about how it works and how the ideas relate to other ideas. But the mathematics you have learned before and know well, such as addition, takes up a small, compact space in your brain. You can use it easily without thinking about it. The process of compression happens because the brain is a highly complex organ with many things to control, and it can focus on only a few uncompressed ideas at any one time. Ideas that are known well are compressed and filed away. William Thurston, a top mathematician who won the Fields Medal, describes compression like this:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through the same process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston, 1990)

Many students do not describe mathematics as a "real joy"—in part because they are not engaging in compression. Notably, the brain can only compress concepts; it cannot compress rules and methods. Therefore students who do not engage in conceptual thinking and instead approach mathematics as a list of rules to remember are not engaging in the critical process of compression, so their brain is unable to organize and file away ideas; instead, it struggles to hold onto long lists of methods and rules. This is why it is so important to help students approach mathematics conceptually at all times. Approaching mathematics conceptually is the essence of what I describe as a mathematical mindset.

How Important Is Math Practice?

When I show parents and teachers the evidence that students need to engage in mathematics conceptually and visually, some parents ask, "But don't students need a lot of math practice?"—by which they mean pages of math questions given in isolation. The question of whether or how much practice students need in mathematics is an interesting one. We know that when learning happens a synapse fires, and in order for structural brain change to happen we need to revisit ideas and learn them deeply. But what does that mean? It is important to revisit mathematical ideas, but the "practice" of methods over and over again is unhelpful. When you learn a new idea in mathematics, it is helpful to reinforce that idea, and the best way to do this is by using it in different ways. We do students a great disservice when we pull out the most simple version of an idea and give students 40 questions that repeat it. Worksheets that repeat the same idea over and over turn students away from math, are unnecessary, and do not prepare them to use the idea in different situations.

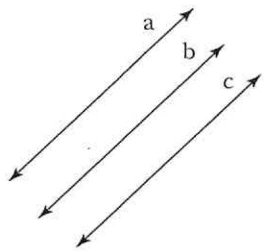
In Malcolm Gladwell's bestselling book *Outliers,* he develops the idea that it takes roughly 10,000 hours of practice to achieve mastery in a field (Gladwell, 2011). Gladwell describes the achievements of famous musicians, chess players, and sports stars, and he shows something important. Many people believe that people such as Beethoven are natural geniuses, but Gladwell shows that they work hard and long to achieve their great accomplishments, with a growth mindset that supports their work. Unfortunately, I have spoken to a number of people who have interpreted Gladwell's idea to mean that students can develop expertise in math after 10,000 hours of mindless practice. This is incorrect. Expertise in mathematics requires 10,000 hours of working mathematically. We do not need students to take a single method and practice it over and over again. That is not mathematics; it does not give students the knowledge of ideas, concepts, and relationships that make up expert mathematics performance. Someone working for 10,000 hours would need to be working on mathematics as a whole, considering mathematical ideas and connections, solving problems, reasoning, and connecting methods.

Most textbook authors in the United States base their whole approach on the idea of isolating methods, reducing them to their simplest form and practicing them. This is problematic for many reasons. First, practicing isolating methods induces boredom in students; many students simply turn off when they think their role is to passively accept a method (Boaler & Greeno, 2000) and repeat it over and over again. Second, most practice examples give the most simplified and disconnected version of the method to be practiced, giving students no sense of when or how they might use the method.

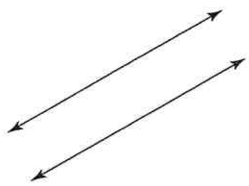
This problem extends to the ways books introduce examples of ideas, as they always give the most simple version. Exhibit 4.1 shows the answers students give to math problems in research studies, highlighting the nature of the problem caused by textbook questions.

EXHIBIT 4.1

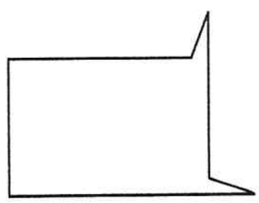
Eleven-year-olds were shown the following figure and asked: Is line a parallel to line c?



Most answered "No, because line b is in the way." This comes about because the concept of parallel lines is almost always illustrated by a picture of two lines like this:



Students were then asked to name the following shape:



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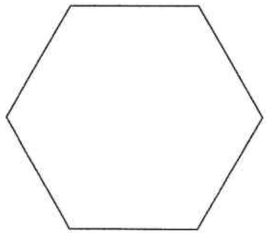
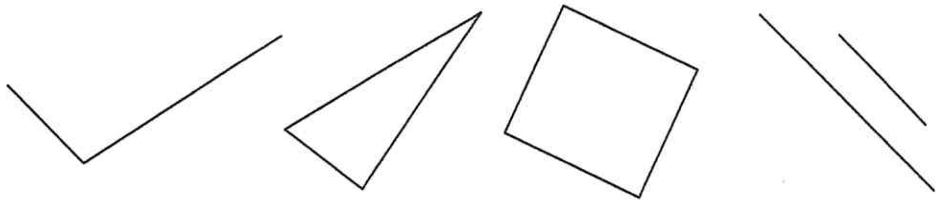
Most were unable to. The shape *is* a hexagon (a six-sided polygon), but hexagons are almost always shown in this way:



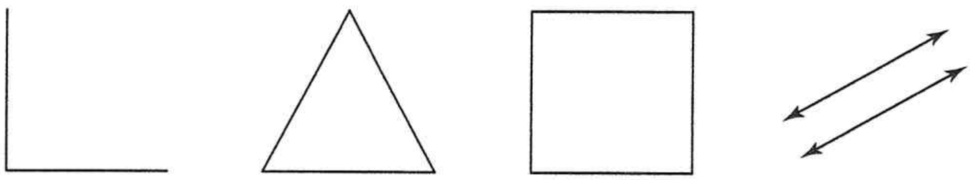
This does not illuminate the full concept of a hexagon well.

Over half of eight-year-olds did not see the following as examples of a right angle, a triangle, a square, or parallel lines . . .



right angle triangle square parallel lines

. . . because they have always been shown the simplest version of the concepts. This is the familiar view that students expect to see:



right angle triangle square parallel lines

The fact that over half of the students in the studies could not name the shapes tells us something important: when textbooks introduce only the simplest version of an idea, students are denied the opportunity to learn what the idea really is. Students were unable to name the different examples because the textbook authors had given "perfect examples" each time. When learning a definition, it is helpful to offer different examples—some of which barely meet the definition and some of which do not meet it at all—instead of perfect examples each time.

Mathematics teachers should also think about the width and breadth of the definition they are showing, and sometimes this is best highlighted by *non-examples*. When learning a definition, it is often very helpful to see both examples that fit the definition and others that *do not* fit, rather than just presenting a series of perfect examples. For example, when learning about birds it can be helpful to think about bats and why they are not birds, rather than to see more and more examples of sparrows and crows.

The misconceptions that students form when shown perfect examples is analogous to the problems students develop when practicing isolated methods over and over. Students are given uncomplicated situations that require the simple use of a procedure (or often, no situation at all). They learn the method, but when they are given realistic mathematics problems or when they need to use math in the world, they are unable to use the methods (Organisation for Economic Co-operation and Development, 2013). Real problems often require the choice and adaptation of methods that students have often never learned to use or even think about. In the next chapter we will look at the nature of rich and effective mathematics problems that avoid these problems.

In an award-winning research study in England, I followed students for three years through a practice approach to mathematics—students were shown isolated examples in math class, which they practiced over and over again (Boaler, 2002a). I contrasted this with an approach in which students were shown the complexity of mathematics and expected to think conceptually at all times, choosing, using, and applying methods. The two approaches were taught in separate schools, to students of the same background and achievement levels, both in low-income areas of the country. The students who were taught to practice methods over and over in a disciplined school in which there were high levels of "time on task" scored at significantly lower levels on the national mathematics examination than students who practiced much less but were encouraged to think conceptually. One significant problem the students from the traditional school faced in the national examination—a set of procedural questions—was that they did not know which method to choose to answer questions. They had practiced methods over and over but had never been asked to consider a situation and choose a method. Here are two of the students from the school reflecting on the difficulties they faced in the exam:

> It's stupid really 'cause when you're in the lesson, when you're doing work—even when it's hard—you get the odd one or two wrong, but most of them you get right and you think, "Well, when I go into the exam, I'm gonna get most of them right," 'cause you get all your chapters right. But you don't. (Alan, Amber Hill)

> It's different, and like the way it's there like—not the same. It doesn't like tell you it—the story, the question; it's not the same as in the books, the way the teacher works it out. (Gary, Amber Hill)

The oversimplification of mathematics and the practice of methods through isolated simplified procedures is part of the reason we have widespread failure in the United States and the United Kingdom. It is also part of the reason that students do not develop mathematical mindsets; they do not see their role as thinking and sense making; rather, they see it as taking methods and repeating them. Students are led to think there is no place for thinking in math class.

In a second study, conducted in the United States, we asked students in a similar practice model of math teaching what their role was in the math classroom (Boaler & Staples, 2005). A stunning 97% of students said the same thing: their role was to "pay careful attention." This passive act of watching—not thinking, reasoning, or sense making—does not lead to understanding or the development of a mathematical mindset.

Students are often given math practice as homework. There is a lot of evidence that homework, of any form, is unnecessary or damaging; I will share some of this evidence in Chapter Six. As a parent, I know that homework is the most common source of tears in our house, and the subject that is most stressful at home is math, especially when math homework consists of nothing but long lists of isolated questions.

Pages of practice problems are sent home—with no thought, it seems, of their negative effect on the home environment that evening. But there is hope: schools that decide to end homework see no reductions in students' achievement and significant increases in the quality of home life (Kohn, 2008).

Large research studies have shown that the presence or absence of homework has minimal or no effects on achievement (Challenge Success, 2012) and that homework leads to significant inequities (Program for International Student Assessment, 2015) (an issue I will return to in Chapter Six), yet homework plays a large negative role in the lives of many parents and children. Research also shows that the only time homework is effective is when students are given a worthwhile learning experience, not worksheets of practice problems, and when homework is seen not as a norm but as an occasional opportunity to offer a meaningful task. My daughters are in schools that know the research on homework and usually assign only worthwhile math homework, such as KENKEN puzzles, but occasionally teachers have sent home 40 questions of practice, on a method such as subtraction or multiplication. I've seen my children's spirits fall when they pull out this kind of homework. In those moments, I explain to them that the page of repeated questions is not what math really is, and after they have shown that they can solve a few of the questions—usually four or five—I suggest they stop. I write a note to the teacher saying that I am satisfied that they have understood the method, and I don't want them working on 35 more questions, as this would give them damaging ideas about the nature of math.

If you are working in a school where homework is required, there are homework problems to give that are much more productive than pages of math practice. Two innovative teachers I work with in Vista Unified School District, Yekaterina Milvidskaia and Tiana Tebelman, developed a set of homework reflection questions that they choose from each day to help their students process and understand the mathematics they have met that day at a deeper level. They typically assign one reflection question for students to respond to each night and one to five mathematical questions to work on (depending on the complexity of the problems). Exhibit 4.2 shows the reflection questions they have developed, from which they choose *one* each night.

Yekaterina and Tiana have used these reflection questions for two years and have noticed a really positive impact on their students, who now reflect on what they have learned in class, synthesize their ideas, and ask more questions in class.

Each year they have given their students a midyear survey to gather data and get their feedback on their classroom practices, including their new homework approach. When they asked students "Please provide us with feedback on your homework format this year," they received the following responses:

> I think that the way we do our homework is very helpful. When you spend more time reflecting about what we learned (written response), and less time doing more math (textbook), you learn a lot more.

> I feel like the homework questions help me reflect on what I learned from the day. If I do not quite remember something, then it gives me a chance to look back into my composition book.

> This year I really like how we do our homework. I understand how to do my homework because of the reflections; those really help me because then I can remember what I did in class that day.

> Having the reflection questions does actually help me a lot. I can see what I need to work on and what I'm doing good on.

The students talk about the ways the questions have helped them in their learning of mathematics. The questions are much less stressful for students, which is always important, and they invite students to think conceptually about big ideas, which is invaluable. Questions that ask students to think about errors or confusions are particularly helpful in encouraging students' self-reflection, and they will often result in the students' understanding the mathematics for the first time. Such questions also give the teacher really important information that can guide their teaching. Similar questions can be written for students at the end of class, as "exit tickets" that they write before they leave the lesson. I will share more ideas for reflection questions in Chapter Eight.

As I mentioned in Chapter One, the PISA team at the Organisation for Economic Co-operation and Development (OECD) not only give mathematics tests to students but also collect data on students' mindset beliefs and mathematics strategies. From looking at the strategies the 13 million students use, the data show that the lowest-scoring students in the world are those who use a memorization strategy. These students prepare for mathematics tests by trying to memorize methods. The highest-scoring students in the world are those who approach mathematics looking at and thinking about the big ideas and the connections between them. Figure 4.4 shows the achievement differences for students who use these different strategies.

One of the very best things we can do for students is to help them develop mathematical mindsets, whereby they believe that mathematics is about thinking, sense making, big ideas, and connections—not about the memorization of methods.

One excellent method for preparing students to think and learn in these ways—appreciating the connected, conceptual nature of mathematics—is a teaching strategy called "number talks."

Math Homework Reflection Questions

Part 1: Written Response Questions

*Your response to the question(s) chosen should be very detailed! Please write in complete sentences and be ready to share your response in class the next day.

1. What were the main mathematical concepts or ideas that you learned today or that we discussed in class today?

2. What questions do you still have about _______________? If you don't have a question, write a similar problem and solve it instead.

3. Describe a mistake or misconception that you or a classmate had in class today. What did you learn from this mistake or misconception?

4. How did you or your group approach today's problem or problem set? Was your approach successful? What did you learn from your approach?

Exhibit 4.2

This is also the very best strategy I know for teaching number sense and math facts at the same time. The method was developed by Ruth Parker and Kathy Richardson. This is an ideal short teaching activity that teachers can start lessons with or parents can use at home. It involves posing an abstract math problem and asking students to show how they solve the problem mentally. The teacher then collects the different methods students give and looks at why they work. For example, a teacher may pose 15 × 12 and find that students solve the problem in these five different ways:

$15 \times 10 = 150$	$30 \times 12 = 360$	$12 \times 15 =$	$12 \times 5 = 60$	$12 \times 12 = 144$
$15 \times 2 = 30$	$360 \div 2 = 180$	6×30	$12 \times 10 = 120$	$12 \times 3 = 36$
$150 + 30 = 180$		$6 \times 30 = 180$	$120 + 60 = 180$	$144 + 36 = 180$

5. Describe in detail how someone else in class approached a problem. How is their approach similar or different to the way you approached the problem?

6. What new vocabulary words or terms were introduced today? What do you believe each new word means? Give an example/picture of each word.

7. What was the big mathematical debate about in class today? What did you learn from the debate?

8. How is ____________ similar or different to ____________?

9. What would happen if you changed ________________?

10. What were some of your strengths and weaknesses in this unit? What is your plan to improve in your areas of weakness?

Exhibit 4.2

Students love to give their different strategies and are usually completely engaged and fascinated by the different methods that emerge. Students learn mental math, they have opportunities to memorize math facts, and they also develop conceptual understanding of numbers and of the arithmetic properties that are critical to success in algebra and beyond. Two books, one by Cathy Humphreys and Ruth Parker (Humphreys & Parker, 2015) and another by Sherry Parrish (Parrish, 2014), illustrate many different number talks to work on with secondary and elementary students, respectively. Number talks are also taught through a video on youcubed, which is an extract from my online teacher/parent course (http://www.youcubed.org/category/teaching-ideas/number-sense/).